NAMIBIA UNIVERSITY
OF SCIENCE AND TECHNOLOGY

FACULTY OF HEALTH AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

QUALIFICATION: Bachelor of Science in Applied Mathematics and Statistics	
QUALIFICATION CODE: 08BSMH	**LEVEL:** 8
COURSE CODE: FAN802S	**COURSE NAME:** FUNCTIONAL ANALYSIS
SESSION: JANUARY 2019	**PAPER:** THEORY
DURATION: 3 HOURS	**MARKS:** 100

SECOND OPPORTUNITY EXAMINATION QUESTION PAPER	
EXAMINER	PROF. G. HEIMBECK
MODERATOR:	PROF. F. MASSAMBA

INSTRUCTIONS

1. Answer ALL the questions in the booklet provided.
2. Show clearly all the steps used in the calculations.
3. All written work must be done in blue or black ink and sketches must be done in pencil.

PERMISSIBLE MATERIALS

1. Non-programmable calculator without a cover.

THIS QUESTION PAPER CONSISTS OF 4 PAGES (Including this front page)

Question 1

a) Let V be a $\mathbb{K}$-vector space.

i) Which subsets of V are linearly independent? State the definition. [4]

ii) Let X be a maximal linearly independent subset of V. Prove that X is a basis of V. [5]

b) State the theorem pertaining to bases of a vector space. [4]

Question 2

Let V be a normed vector space over $\mathbb{K}$.

a) If $(a_n)_{\mathbb{N}}$ is a convergent sequence in X, prove that the limit of $(a_n)_{\mathbb{N}}$ belongs to $\overline{X}$. [4]

b) Let $c \in \overline{X}$. Prove that there exists a sequence in X which converges to c. [6]

c) Prove that the closure of a subspace of V is a subspace of V. [6]

Question 3

Let V be a normed $\mathbb{K}$-vector space.

a) Let $(a_n)_{\mathbb{N}}$ be a sequence in V. Prove that $(a_n)_{\mathbb{N}}$ is a Cauchy sequence in V if and only if for each $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ such that $\|a_n - a_N\| < \varepsilon$ for all $n \geq N$. [4]

b) Let $\sum a_k$ be an absolutely convergent series in V.

i) Prove that $\sum a_k$ is a Cauchy sequence. [6]

ii) Is $\sum a_k$ convergent? Explain your answer. [4]

Question 4

Let V be a normed $\mathbb{K}$-vector space and $\varphi : V \to \mathbb{K}$ a linear form such that $\varphi \neq 0$.

a) Prove that $H := \ker\varphi$ is a hyperplane of V. [5]

b) Show that H is closed or dense. [4]

c) If H is dense, show that φ is not continuous. [6]

Question 5

Let V and W be normed vector spaces over the same field $\mathbb{K}$ and let $f : V \to W$ be a continuous linear mapping.

a) What is $\|f\|$ by definition? If $\|f\| = 0$, show that $f = 0$. [5]

b) Prove that $\|f(x)\| \leq \|f\|\|x\|$ for all $x \in V$ [4]

c) Show that f is Lipschitz continuous. [3]

Question 6

a) What is a Hermitian vector space? State the definition. [4]

b) Let (V, Φ) be a Hermitian vector space.

   i) State and prove the Cauchy-Schwarz inequality. [6]

   ii) Show that

$$\|x\| := \sqrt{\Phi(x, x)}$$

   defines a norm on V. [5]

Question 7

a) Let V, W be normed vector spaces over the same field $\mathbb{K}$ and let $f : V \to W$ be a continuous linear mapping. Let U be a subspace of V and let g denote the restriction of f to U. How are $\|g\|$ and $\|f\|$ related? Explain! [5]

b) State the theorem of Hahn-Banach. [4]

c) Let V be a normed vector space and $a \in V - \{0\}$. Prove that there exists a closed hyperplane H of V such that $[a] \oplus H = V$. [6]

End of the question paper.